LIE GROUPOIDS AND GENERALIZED ALMOST SUBTANGENT MANIFOLDS

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Abstract. In this paper, we show that there is a close relationship between generalized subtangent manifolds and Lie groupoids. We obtain equivalent assertions among the integrability conditions of generalized almost subtangent manifolds, the condition of compatibility of source and target maps of symplectic groupoids with symplectic form and generalized subtangent maps.

1. Introduction

A groupoid is a small category in which all morphisms are invertible. More precisely, a groupoid G consists of two sets G_1 and G_0, called arrows and the objects, respectively, with maps $s, t : G_1 \to G_0$ called source and target. It is equipped with a composition $m : G_2 \to G_1$ defined on the subset $G_2 = \{(g, h) \in G_1 \times G_1 | s(g) = t(h)\}$; an inclusion map of objects $e : G_0 \to G_1$ and an inversion map $i : G_1 \to G_1$. For a groupoid, the following properties are satisfied: $s(gh) = s(h)$, $t(gh) = t(g)$, $s(g^{-1}) = t(g)$, $t(g^{-1}) = s(g)$, $g(hf) = (gh)f$ whenever both sides are defined, $g^{-1}g = 1_{s(g)}$, $gg^{-1} = 1_{t(g)}$. Here we have used $gh, 1_x$ and g^{-1} instead of $m(g, h)$, $e(x)$ and $i(g)$. Generally, a groupoid G is denoted by the set of arrows G_1.

A topological groupoid is a groupoid G_1 whose set of arrows and set of objects are both topological spaces whose structure maps s, t, e, i, m are all continuous and s, t are open maps.

A Lie groupoid is a groupoid G whose set of arrows and set of objects are both manifolds whose structure maps s, t, e, i, m are all smooth maps and s, t are submersions. The latter condition ensures that s and t-fibres are manifolds. One can see from above definition the space G_2 of composable arrows is a submanifold of $G_1 \times G_1$. We note that Lie groupoid introduced by Ehresmann [1].

On the other hand, Lie algebroids were first introduced by Pradines [2] as infinitesimal objects associated with the Lie grou- poids. More precisely, a Lie algebroid structure on a real vector bundle A on a manifold M is defined by a vector bundle map $\rho_A : A \to TM$, the anchor of A, and an $\mathbb{R}$-Lie algebra bracket on $\Gamma(A), [,]_A$ satisfying the Leibnitz rule

$$[\alpha, f\beta]_A = f[\alpha, \beta]_A + L_{\rho_A(\alpha)}(f)\beta$$

for all $\alpha, \beta \in \Gamma(A), f \in C^\infty(M)$, where $L_{\rho_A(\alpha)}$ is the Lie derivative with respect to the vector field $\rho_A(\alpha)$. And $\Gamma(A)$ denotes the set of sections in A.

On the other hand, Hitchin [3] introduced the notion of generalized complex manifolds by unifying and extending the usual notions of complex and symplectic manifolds. Later the notion of generalized Kähler manifold was introduced by Gualtieri [4] and submanifolds of such manifolds have been studied in many papers.

As an analogue of generalized complex structures on even dimensional manifolds, the concept of generalized almost subtangent manifolds were introduced in [5] and such manifolds have been studied in [6] and [5].

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Recently, Crainic [7] showed that there is a close relationship between the equations of a generalized complex manifold and a Lie groupoid. More precisely, he obtained that the complicated equations of such manifolds turn into simple structures for Lie groupoids.

In this paper, we investigate relationships between the complicated equations of generalized subtangent structures and Lie groupoids. We showed that the equations of such manifolds are useful to obtain equivalent results on a symplectic groupoid.

2. Preliminaries

In this section we recall basic facts of Poisson geometry, Lie groupoids and Lie algebroids. More details can be found in [8] and [9]. A central idea in generalized geometry is that $TM \oplus T^*M$ should be thought of as a generalized tangent bundle to manifold M. If X and ξ denote a vector field and a dual vector field on M respectively, then we write (X, ξ) (or $X+\xi$) as a typical element of $TM \oplus T^*M$. The Courant bracket of two sections $(X, \xi), (Y, \eta)$ of $TM \oplus T^*M = \mathcal{TM}$ is defined by

$$\begin{aligned} [(X,\xi),(Y,\eta)] &= [X,Y] + L_X\eta - L_Y\xi \\ &\quad -\frac{1}{2}d(i_X\eta - i_Y\xi), \end{aligned} \tag{2.1}$$

where d, L_X and i_X denote exterior derivative, Lie derivative and interior derivative with respect to X, respectively. The Courant bracket is antisymmetric but, it does not satisfy the Jacobi identity. We adapt the notions $\beta(\pi^\sharp\alpha) = \pi(\alpha, \beta)$ and $\omega_\sharp(X)(Y) = \omega(X, Y)$ which are defined as $\pi^\sharp : T^*M \to TM$, $\omega_\sharp : TM \to T^*M$ for any 1-forms α and β, 2-form ω and bivector field π, and vector fields X and Y. Also we denote by $[,]_\pi$, the bracket on the space of 1-forms on M defined by

$$[\alpha, \beta]_\pi = L_{\pi^\sharp\alpha}\beta - L_{\pi^\sharp\beta}\alpha - d\pi(\alpha, \beta).$$

On the other hand, a symplectic manifold is a smooth (even dimensional) manifold M with a non-degenerate closed 2-form $\omega \in \Omega^2(M)$. ω is called the symplectic form of M. Let G be a Lie groupoid on M and ω a form on Lie groupoid G, then ω is called multiplicative if

$$m^*\omega = pr_1^*\omega + pr_2^*\omega,$$

where $pr_i : G \times G \to G$, $i = 1, 2$, are the canonical projections.

We now recall the notion of Poisson manifolds. A Poisson manifold is a smooth manifold M whose function space $C^\infty(M, \mathbb{R})$, is a Lie algebra with bracket $\{,\}$, such that the following properties are satisfied;

(i) $\{f, g\} = -\{g, f\}$
(ii) $\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$
(iii) $\{fg, h\} = f\{g, h\} + g\{f, h\}$.

If M is a Poisson manifold, then there is a unique bivector π, called the Poisson bivector, and a unique homomorfizm $\pi^\sharp : T^*M \to TM$ of vector bundles with $\pi^\sharp(T^*M) \subset TM$ such that $\pi(df, dg) = \pi^\sharp(df)g = \{f, g\}$.

It is also possible to define a Poisson manifold by using the bivector π. Indeed, a smooth manifold is a Poisson manifold if $[\pi, \pi] = 0$, where $[,]$ denotes the Schouten bracket on the space of multivector fields.

We now give a relation between Lie algebroid and Lie groupoi- d. Given a Lie groupoid G on M, the associated Lie algebroid $A = Lie(G)$ has fibres $A_x = Ker(ds)_x = T_x(G(-, x))$, for

any $x \in M$. Any $\alpha \in \Gamma(A)$ extends to a unique right-invariant vector field on G, which will be denoted by same letter α. The usual Lie bracket on vector fields induces the bracket on $\Gamma(A)$, and the anchor is given by $\rho = dt : A \to TM$.

Given a Lie algebroid A, an integration of A is a Lie groupoid G together with an isomorphism $A \cong Lie(G)$. If such a G exists, then it is called that A is integrable. In contrast with the case of Lie algebras, not every Lie algebroid admits an integration. However if a Lie algebroid is integrable, then there exists a canonical source simply connected integration G, and any other source simply connected integration is smoothly isomorphic to G. *From now on we assume that all Lie groupoids are source-simply-connected.*

In this section, finally, we recall the notion of IM form (infinitesimal multiplicative form) on a Lie algebroid [10]. More precisely, an IM form on a Lie algebroid A is a bundle map

$$u : A \to T^*M$$

satisfying the following properties

(i) $\langle u(\alpha), \rho(\beta)\rangle = -\langle u(\beta), \rho(\alpha)\rangle$
(ii) $u([\alpha, \beta]) = L_\alpha(u(\beta)) - L_\beta(u(\alpha))$
$+d\langle u(\alpha), \rho(\beta)\rangle$

for $\alpha, \beta \in \Gamma(A)$, where $\langle, \rangle$ denotes the usual pairing between a vector space and its dual.

If A is a Lie algebroid of a Lie groupoid G, then a closed multiplicative 2-form ω on G induces an IM form u_ω of A by

$$\langle u_\omega(\alpha), X\rangle = \omega(\alpha, X).$$

For the relationship between IM form and closed 2-form we have the following.

Theorem 1. [10]*If A is an integrable Lie algebroid and if G is its integration, then $\omega \mapsto u_\omega$ is an one to one correspondence between closed multiplicative 2-forms on G and IM forms of A.*

If a Lie groupoid G is endowed with a symplectic form which is multiplicative, then G is called symplectic groupoid.

Similar to 2-forms, given a Lie groupoid G, a (1,1)-tensor $J : TG \to TG$ is called multiplicative [7] if for any $(g, h) \in G \times G$ and any $v_g \in T_gG$, $w_h \in T_hG$ such that (v_g, w_h) is tangent to $G \times G$ at (g, h), so is (Jv_g, Jw_h), and

$$(dm)_{g,h}(Jv_g, Jw_h) = J((dm)_{g,h}(v_g, w_h)).$$

3. Lie Groupoids and Generalized Subtangent Structures

In this section we first give a characterization for generalized subtangent manifolds, then we obtain certain relationships between generalized subtangent manifolds and symplectic groupoids. We recall that a generalized almost subtangent structure $\mathcal{J}$ is an endomorphism on $\mathcal{TM}$ such that $\mathcal{J}^2 = 0$. A generalized almost subtangent structure can be represented by classical tensor fields as follows:

$$\mathcal{J} = \begin{bmatrix} a & \pi^\sharp \\ \sigma_\sharp & -a^* \end{bmatrix} \tag{3.1}$$

where π is a bivector on M, σ is a 2-form on M, $a : TM \to TM$ is a bundle map, and $a^* : T^*M \to T^*M$ is dual of a, for almost subtangent structures see:[5] and [6].

A generalized almost subtangent structure is called integrable (or just subtangent structure) if $\mathcal{J}$ satisfies the following condition

$$[\mathcal{J}\alpha, \mathcal{J}\beta] - \mathcal{J}([\mathcal{J}\alpha, \beta] + [\alpha, \mathcal{J}\beta]) = 0, \tag{3.2}$$

for all sections $\alpha, \beta \in \mathcal{TM}$.

In the sequel, we give necessary and sufficient conditions for a generalized almost subtangent structure to be integrable in terms of the above tensor fields. We note that the following result was stated in [5], but its proof was not given in there. In fact, the proof of the conditions of the following proposition is similar to the proposition given in [7] by Crainic for generalized complex structures. Although the conditions are similar to the generalized complex case, their proofs are slightly different from the complex case. Therefore we give one part of the proof of the following proposition.

Proposition 1. *A manifold with $\mathcal{J}$ given by (3.1) is a generalized subtangent manifold if and only if*

(S1) *π satisfies the equation*

$$\pi^\sharp([\xi,\eta]_\pi) = [\pi^\sharp(\xi), \pi^\sharp(\eta)],$$

(S2) *π and a are related by the following two formulas*

$$\begin{aligned} a\pi^\sharp &= \pi^\sharp a^*, \\ a^*([\xi,\eta]_\pi) &= L_{\pi^\sharp\xi}(a^*\eta) - L_{\pi^\sharp\eta}(a^*\xi) \\ &\quad -d\pi(a^*\xi,\eta), \end{aligned} \tag{3.3}$$

(S3) *π, a and σ are related by the following two formulas*

$$a^2 + \pi^\sharp\sigma_\sharp = 0, \tag{3.4}$$

$$N_a(X,Y) = \pi^\sharp(i_{X\wedge Y}d(\sigma)), \tag{3.5}$$

(S4) *σ and a are related by the following two formulas*

$$\begin{aligned} a^*\sigma_\sharp &= \sigma_\sharp a, \\ d\sigma_a(X,Y,Z) &= d\sigma(aX,Y,Z) + d\sigma(X,aY,Z) \\ &\quad +d\sigma(X,Y,aZ) \end{aligned}$$

for all 1-forms ξ and η, and all vector fields X, Y and Z, where $\sigma_a(X,Y) = \sigma(aX,Y)$.

As an analogue of a Hitchin pair on generalized complex manifold, a Hitchin pair on a generalized almost subtangent manifold M is a pair (ω, a) consisting of a symplectic form ω and a (1,1)-tensor a with the property that ω and a commute (i.e $\omega(X,aY) = \omega(aX,Y)$) and $d\omega_a = 0$, where $\omega_a(X,Y) = \omega(aX,Y)$.

Lemma 1. *If π is a non-degenerate bivector on a generalized almost subtangent manifold M, ω is the inverse 2-form (defined by $\omega_\sharp = (\pi^\sharp)^{-1}$) and π satisfies (3.4) then $\sigma = -a^*\omega$.*

Proof. For $X \in \chi(M)$, we apply $\omega_\sharp$ to (3.4) and using the dual subtangent structure a^*, we have

$$(a^*)^2\omega_\sharp(X) + \sigma_\sharp(X) = 0.$$

Now for $Y \in \chi(M)$, since ω and a are commute, we obtain

$$\omega(aX,aY) + \sigma(X,Y) = 0.$$

Thus we get

$$a^*\omega(X,Y) + \sigma(X,Y) = 0. \tag{3.6}$$

Since the equation (3.6) is hold for all X and Y, we get

$$\sigma = -a^*\omega.$$

□

We say that 2-form σ is the twist of Hitchin pair (ω, a).

A symplectic+subtangent structure on a generalized almost subtangent manifold M consists of a pair (ω, J) with ω-symplecti- c and J-subtangent structure on M, which commute.

Lemma 2. *Let (M, ω) be a symplectic manifold. (ω, a) is a symplectic+subtangent structure if and only if $d\omega_a = 0$, $a^*\omega = 0$.*

Proof. We will only prove the sufficient condition. Since (M, ω) is a symplectic manifold, then $d\omega = 0$. Since $d\omega_a = 0$, $a^*\omega = 0$, by using the following equation (see [7]),

$$\begin{aligned} i_{N_a(X,Y)}(\omega) &= i_{aX\wedge Y + X\wedge aY}(d\omega_a) - i_{aX\wedge aY}(d\omega) \\ &\quad - i_{X\wedge Y}(d(a^*\omega)), \end{aligned} \tag{3.7}$$

we get $i_{N_a(X,Y)}(\omega) = -i_{X\wedge Y}(d(a^*\omega)) = 0$. Hence,

$$\omega(N_a(X, Y), \bullet) = 0.$$

Since ω is non-degenerate, then $N_a = 0$. Thus a is a subtangent structure. On the other hand, $a^*\omega = 0$ implies that ω and a commute.

The converse is clear. □

Next we relate (S1) and the 2-form ω.

Lemma 3. *If π is a non-degenerate bivector on a generalized almost subtangent manifold M, and ω is the inverse 2-form (defined by $\omega_\sharp = (\pi^\sharp)^{-1}$), then π satisfies (S1) if and only if ω is closed.*

Proof. Applying $\xi = i_X(\omega)$ and $\eta = i_Y(\omega)$ to (S1), we get

$$\pi^\sharp(L_X(i_Y(\omega)) - L_Y(i_X(\omega)) - d(\omega_\sharp Y(\pi^\sharp\omega_\sharp X)) = [X, Y]$$

Since $\omega_\sharp = (\pi^\sharp)^{-1}$, we have

$$\pi^\sharp(L_X(i_Y(\omega)) - L_Y(i_X(\omega)) - d(\omega_\sharp Y)(X)) = [X, Y] \tag{3.8}$$

Then applying $\omega_\sharp$ to (3.8), we derive

$$L_X(i_Y(\omega)) - L_Y(i_X(\omega)) - d(\omega_\sharp Y)(X) = \omega_\sharp([X, Y])$$

Using

$$\begin{aligned} i_{X\wedge Y}(d\sigma) &= L_X(i_Y\sigma) - L_Y(i_X\sigma) + d(i_{X\wedge Y}\sigma) \\ &\quad - i_{[X,Y]}\sigma, \end{aligned} \tag{3.9}$$

formula, then we get

$$L_X(i_Y\omega) - L_Y(i_X\omega) + d(i_{X\wedge Y}\omega) - i_{[X,Y]}\omega = i_{X\wedge Y}d\omega.$$

Since left hand side vanishes in above equation, then $i_{X\wedge Y}d\omega = 0$. Thus $d\omega = 0$. The converse is clear. □

Thus, we have the following result which shows that there is close relationship between condition (S1) and a symplectic groupoid.

Theorem 2. *Let M be a generalized almost subtangent manifold. There is a 1-1 correspondence between:*

(i) *Integrable bivectors π on M satisfying (S1),*
(ii) *Symplectic groupoids (Σ, ω) over M.*

Since $\pi^\sharp$ and $[,]_\pi$ define a Lie algebroid structure on T^*M, one can obtain the above theorem by following the steps given in ([7],Theorem 3.2). We now give the conditions for (S2) in terms of ω and ω_a.

Lemma 4. *Let M be a generalized almost subtangent manifold and ω a symplectic form. Given a non-degenerate bivector π (i.e. $\pi^\sharp = (\omega_\sharp)^{-1}$) and a map $a : TM \to TM$, then π and a satisfy (S2) if and only if ω and a commute and ω_a is closed.*

Proof. For a 1-form ξ, we use $\xi = i_X\omega = \omega_\sharp X$ such that X is an arbitrary vector field. Since

$$a\pi^\sharp(i_X\omega) \;=\; \pi^\sharp a^*(i_X\omega), \tag{3.10}$$

applying $\omega_\sharp$ to (3.10) and using $\pi^\sharp = (\omega_\sharp)^{-1}$; for a vector field Y, we have

$$\omega(aX, Y) = a^*(i_X\omega)(Y),$$

which gives

$$\omega(aX, Y) \;=\; \omega(X, aY).$$

Let ξ and η be 1-forms such that $\xi = i_X\omega = \omega_\sharp(X)$ and $\eta = i_Y\omega = \omega_\sharp(Y)$ for arbitrary vector fields X and Y. Then from (3.3) we have

$$\begin{aligned} a^*(L_{\pi^\sharp\omega_\sharp(X)}(\omega_\sharp(Y)) - L_{\pi^\sharp\omega_\sharp(Y)}(\omega_\sharp(X)) - d\pi(i_X\omega, i_Y\omega)) & \\ = L_{\pi^\sharp\omega_\sharp(X)}(a^*\omega_\sharp(Y)) & \\ -L_{\pi^\sharp\omega_\sharp(Y)}(a^*\omega_\sharp(X)) & \\ -d\pi(a^*i_X\omega, i_Y\omega). & \end{aligned}$$

Since $\pi^\sharp = (\omega_\sharp)^{-1}$ and $a^*i_Y\omega = i_Y\omega_a$, we obtain

$$\begin{aligned} a^*(L_X(i_Y\omega) - L_Y(i_X\omega) - d(i_{Y\wedge X}\omega)) &= L_X(i_Y\omega_a) \\ &- L_Y(i_X\omega_a) \\ &- d(i_Y\omega(\pi^\sharp(\omega_\sharp(aX)))). \end{aligned}$$

Using (3.9) for left hand side, then we get

$$a^*(i_{X\wedge Y}(d\omega) + i_{[X,Y]}\omega) = L_X(i_Y\omega_a) - L_Y(i_X\omega_a) - d(\omega(Y, aX)).$$

Since ω and a commute, using also this for the right hand side, we have

$$a^*(i_{X\wedge Y}(d\omega)) + i_{[X,Y]}\omega_a = i_{X\wedge Y}(d\omega_a) + i_{[X,Y]}\omega_a. \tag{3.11}$$

Since ω is closed, (3.11) implies that $i_{X\wedge Y}(d\omega_a) = 0$, i.e. ω_a is closed. □

Note that it is well known that there is one to one correspondence between (1,1)-tensors a commuting with ω and 2-forms on M. On the other hand, it is easy to see that (S2) is equivalent to the fact that a^* is an IM form on the Lie algebroid T^*M associated Poisson structure π. Thus from the above discussion, Lemma 4 and Theorem 1, one can conclude the following theorem.

Theorem 3. *Let M be a generalized almost subtangent manifold. Let π be an integrable Poisson structure on M, and (Σ, ω) the symplectic groupoid over M. Then there is a natural 1-1 correspondence between*

(i) *(1,1)-tensors a on M satisfying (S2).*
(ii) *multiplicative (1,1)-tensors J on Σ with the property that (J, ω) is a Hitchin pair.*

We recall the notion of generalized subtangent map between generalized subtangent manifolds. This notion was given in [5] similar to the generalized subtangent map between generalized complex manifolds given in [7].

Let $(M_i, \mathcal{J}_i)$, $i = 1, 2$, be two generalized subtangent manifolds, and let a_i, π_i, σ_i be the components of $\mathcal{J}_i$ in the matrix representation (3.1). A map $f : M_1 \to M_2$ is called generalized subtangent iff f maps π_1 into π_2, $f^*\sigma_2 = \sigma_1$ and $(df) \circ a_1 = a_2 \circ (df)$[5].

We now state and prove the main result of this paper. This result gives equivalent assertions between the condition (S3), twist σ of (ω, J) and subtangent maps for a symplectic groupoid over M.

Theorem 4. *Let M be a generalized almost subtangent manifold and (Σ, ω, J) the induced symplectic groupoid over M with the induced multiplicative (1,1)-tensor. Assume that (π, J) satisfy (S1), (S2) with integrable π. Then for a 2-form on M, the following assertions are equivalent.*

(i) *(S3) is satisfied,*
(ii) $-J^*\omega = t^*\sigma - s^*\sigma$,
(iii) $(t, s) : \Sigma \to M \times \overline{M}$ *is generalized subtangent map; (condition of generalized subtangent map on M is $(dt) \circ a_1 = a_2 \circ (dt)$, this condition on $\overline{M}$ is $(ds) \circ a_1 = -a_2 \circ (ds)$).*

Proof. (i)⇔ (ii). Define $\phi = \widetilde{\sigma} - t^*\sigma + s^*\sigma$, such that $\widetilde{\sigma} = -J^*\omega$ and $A = ker(ds)|_M$. We know from Theorem 1 that closed multiplicative 2-form θ on Σ vanishes if and only if IM form $u_\theta = 0$, i.e. $\theta(X, \alpha) = 0$, such that $X \in TM$, $\alpha \in A$. This case can be applied for forms with high dimension, i.e. 3-form θ vanishes if and only if $\theta(X, Y, \alpha) = 0$.

Since ω and ω_J are closed, from (3.7) we get $i_{X\wedge Y}(d(J^*\omega)) = -i_{N_J(X,Y)}\omega$. Putting $\widetilde{\sigma} = -J^*\omega$, we obtain

$$i_{X\wedge Y}(d\widetilde{\sigma}) = i_{N_J(X,Y)}\omega. \tag{3.12}$$

Since $d\phi = 0 \Leftrightarrow d\phi(X, Y, \alpha) = 0$, we have

$$\begin{aligned} d\phi(X, Y, \alpha) = 0 \Leftrightarrow d\widetilde{\sigma}(X, Y, \alpha) &- d(t^*\sigma)(X, Y, \alpha) \\ &+ d(s^*\sigma)(X, Y, \alpha) = 0 \end{aligned}$$

On the other hand, we obtain

$$d(t^*\sigma)(X, Y, \alpha) = d\sigma(dt(X), dt(Y), dt(\alpha)). \tag{3.13}$$

If we take $dt = \rho$ in (3.13) for A, we get

$$d(t^*\sigma)(X, Y, \alpha) = d\sigma(dt(X), dt(Y), \rho(\alpha)). \tag{3.14}$$

On the other hand, from [10] we know that

$$Id_\Sigma = m \circ (t, Id_\Sigma). \tag{3.15}$$

Differentiating (3.15), we obtain

$$X = dt(X). \tag{3.16}$$

Using (3.16) in (3.14), we get

$$d(t^*\sigma)(X, Y, \alpha) = d\sigma(X, Y, \rho(\alpha)).$$

In a similar way, we see that

$$d(s^*\sigma)(X, Y, \alpha) = d\sigma(ds(X), ds(Y), ds(\alpha)).$$

Since $\alpha \in kerds$, then $ds(\alpha) = 0$. Hence $d(s^*\sigma) = 0$.
Thus

$$d\widetilde{\sigma}(X, Y, \alpha) = d\sigma(X, Y, \rho(\alpha)). \tag{3.17}$$

Using (3.12) in (3.17), we derive

$$\omega(N_J(X,Y),\alpha) = d\sigma(X,Y,\rho(\alpha)). \tag{3.18}$$

On the other hand, it is clear that $\phi = 0 \Leftrightarrow \widetilde{\sigma} - t^*\sigma + s^*\sigma = 0$. Thus we obtain

$$\widetilde{\sigma}(X,\alpha) = \sigma(X,\rho(\alpha)).$$

Since $\widetilde{\sigma} = -J^*\omega$, we get

$$-\,\omega(JX,J\alpha) = \sigma(X,\rho(\alpha)). \tag{3.19}$$

Since Poisson bivector π is integrable, it defines a Lie algebroid whose anchor map is $\rho = \pi^\sharp$. Let us use $\pi^\sharp$ instead of ρ in (3.18) and (3.19). Then we get

$$\omega(N_J(X,Y),\alpha) = d\sigma(X,Y,\pi^\sharp(\alpha)), \tag{3.20}$$

$$-\,\omega(JX,J\alpha) = \sigma(X,\pi^\sharp(\alpha)).$$

Since $\omega(\alpha,X) = \alpha(X)$, $\omega_J(\alpha,X) = \alpha(JX)$, from (3.20) we have

$$\begin{aligned} -\alpha(N_J(X,Y)) &= d\sigma(X,Y,\pi^\sharp(\alpha)) \\ &= i_{X\wedge Y}d\sigma(\pi^\sharp(\alpha)) \\ &= \pi(\alpha, i_{X\wedge Y}d\sigma) \\ &= -\alpha(\pi^\sharp(i_{X\wedge Y}d\sigma)). \end{aligned}$$

i.e. $\alpha(N_J(X,Y)) = \alpha(\pi^\sharp(i_{X\wedge Y}d\sigma))$.
Since above equation is hold for all non-degenerate α, we get

$$N_J(X,Y) = \pi^\sharp(i_{X\wedge Y}d\sigma). \tag{3.21}$$

On the other hand, from (3.19) we obtain

$$\begin{aligned} \alpha(a^2X) &= i_X\sigma(\pi^\sharp(\alpha)) \\ &= \pi(\alpha, i_X\sigma) \\ &= -\alpha(\pi^\sharp\sigma_\sharp X). \end{aligned}$$

Thus we get

$$a^2 + \pi^\sharp\sigma_\sharp = 0. \tag{3.22}$$

Then (i)$\Leftrightarrow$(ii) follows from (3.21) and (3.22).

(ii)$\Leftrightarrow$(iii) $-J^*\omega = t^*\sigma - s^*\sigma$ says (t,s) is compatiple with 2-forms. Also it is clear that (t,s) and bivectors are compatible due to Σ is a symplectic groupoid. We will check the compatibility of (t,s) and (1,1)-tensors. From compatibility condition of t and s, we will get $dt \circ J = a \circ dt$ and $ds \circ J = -a \circ ds$.
For all $\alpha \in A$ and $V \in \chi(\Sigma)$,

$$\omega(\alpha, V) = \omega(\alpha, dtV)$$

which is equivalent to

$$\alpha(V) = \langle u_\omega(\alpha), dtV\rangle.$$

Since $u_\omega = Id$ and $u_{\omega_J} = a^*$, we get

$$\begin{aligned} \langle \alpha, a(dt(V))\rangle &= \alpha(a(dt(V))) \\ &= \omega(\alpha, dt(JV)) \\ &= \langle \alpha, dt(JV)\rangle. \end{aligned}$$

Since this equation is hold for all α, then $a(dt) = dt(J)$. Using $s = t \circ i$,

$$\begin{aligned} a(ds(V)) &= ad(t \circ i)V \\ &= -ds(JV), \end{aligned}$$

which shows that $a(ds) = -ds(J)$. Thus proof is completed. □

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